STABILITY OF FLOWS WITH VORTEX DISCONTINUITY

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The solution to flows with a vortex discontinuity is widely used in hydrodynamics of ideal fluids. They occur, e.g., in problems involving the matching of potential and vortical flows [1]. The present paper considers the stability of such flows in a plane. The integral of linearized equations of motion has been obtained in a quadratic form for the disturbance velocity field, vortex, and normal displacement of the surface of discontinuity. Conditions for a positive determinancy of this form lead to sufficient conditions for stability in rms terms, extending the known solutions [2-4] for flows with continuous vorticity. Examples are given for stable flows including flows in curvilinear slot, plane-parallel, and circular flows. Conditions for nonlinear instability are given for the latter two types of flows with piecewise constant vorticity.

1. Basic Flow and Class of Disturbances. Plane flows of an ideal, incompressible, homogeneous fluid are considered in the region τ with a stationary impermeable boundary $\partial\tau$. The results are valid for regions τ of sufficiently general type, though, in particular, we consider a curvilinear ring (circular slot), whose boundary is comprised of closed contours R_+ and R_-. In cartesian coordinates x, y, the stationary flow field has been specified with x- and y-components of velocity, stream function, vorticity, and pressure:

$$U(\mathbf{x}),\ V(\mathbf{x}),\ \Psi(\mathbf{x}),\ \Omega(\mathbf{x}),\ P(\mathbf{x}),\ \mathbf{x}\equiv(x,\ y),\ U=-\Psi_y,\quad V=\Psi_x,\ \Omega\equiv V_x-U_y. \tag{1.1}$$

The subscript with independent variables denotes partial derivatives. It is assumed that U and V are continuous, and their first and second derivatives are continuous almost everywhere in τ except R determined by the curves where Ω has a finite discontinuity. In particular, the flow with closed streamlines ψ = const is investigated, with each streamline enclosing the inner boundary of the ring. The only closed contour of the discontinuity R is one of the lines ψ = const and divides τ into two rings: τ_+ and τ_-. The positive sign denotes the region that is to the left as one moves along R in the direction of the velocity vector. The superposition of disturbances transposes the velocity field $\mathbf{U}=(U,\ V)$, pressure P, regions $\tau_\pm$, and their contact boundary R into $\mathbf{u}^*=(u^*,\ v^*)$, p^*, $\tau_\pm^*$ and R^*. The analysis of the stability problems is carried out with the technique adopted from problems on two-phase flows: independent solutions are obtained in the regions $\tau_\pm^*$ with conditions for their equality at the contact line R^*. In accordance with this field (1.1), $\mathbf{u}^*$ and p^* are specified independently in τ_+, τ_- and τ_+^*, τ_-^*, which is implied without the explicit introduction of $U_\pm$, $u_\pm^*$ etc.

In $\tau_\pm$, functions (1.1) satisfy equations

$$\Omega V=H_x,\ \Omega U=-H_y,\ U_x+V_y=0,\quad H\equiv P+Q^2/2,\ Q^2\equiv U^2+V^2, \tag{1.2}$$

which lead to the presence of functional relations $\Omega=\Omega(\Psi)$. The no-slip condition is satisfied on $R_\pm$

$$\mathbf{U}\cdot\mathbf{n}=0 \tag{1.3}$$

($\mathbf{n}$ is normal to $\partial\tau$). Using square brackets to denote discontinuity of hydrodynamic fields on R,

$$[U]=[V]=[P]=[\nabla P]=[\Psi]=0,\quad [\Omega]\equiv\Omega_+-\Omega_-\neq 0,\ [\Omega']\neq 0,\ \Omega'\equiv d\Omega/d\Psi. \tag{1.4}$$

The class of disturbances, in which the stability problem is being investigated, is selected on the basis of the same requirements of smoothness which is satisfied by the basic flow (1.1). It is assumed that the total velocity fields $\mathbf{u}^*$ are continuous and have continuous first and second derivatives in $\tau_\pm^*$. Only the continuity of the velocity field $\mathbf{u}^*$

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is required on the moving curves R^*. It thus implies that if the initial conditions for the velocity $\mathbf{u}^*$ are chosen to be continuous, then there will be no contact discontinuity at any given time [5]. The continuity condition on both the velocity components on R^* is somewhat stronger than the kinematic and dynamic conditions: it coincides with them for the given class of continuous fields $\mathbf{u}^*$. Thus, we assume

$$[u^*]^* = [v^*]^* = 0, \tag{1.5}$$

where the star outside the bracket denotes that the discontinuity is computed on R^*.

2. Lagrangian Displacement and Linearization Procedure. Linearization in problems with an unknown moving boundary R^* is often carried out in Eulerian system with the "transposal" of boundary conditions from R^* to R. Difficulties then arise in following the very Eulerian disturbance fields in the region between R^* and R, as well as in the interpretation of the "transposal" procedure. A linearization technique [6] free from these difficulties is described here. It is based on the consideration of a Lagrangian description of the disturbed flow with a subsequent replacement of Lagrangian system of coordinates by Eulerian system of coordinates for the fluid particled in the undisturbed flow.

Consider separately, fluid occupying the region τ_+, $\overset{*}{\tau}_+$. Let the undisturbed flow (1.1) of fluid particles be described by the relation between the Eulerian $\mathbf{x}$ and Lagrangian $\mathbf{a}$ coordinates:

$$\mathbf{x} = \mathbf{x}(\mathbf{a},\ t),\ \mathbf{a} = \mathbf{x}(\mathbf{a},\ 0),\ \mathbf{U}(\mathbf{a},\ t) \equiv \partial\mathbf{x}(\mathbf{a},\ t)/\partial t \tag{2.1}$$

($\mathbf{x}$, as well as $\mathbf{a}$, are determined in τ_+). After the superposition of disturbances, the same fluid occupies another region $\overset{*}{\tau}_+$, and its motion is described by other functions of the same variables $\mathbf{a}$, t:

$$\mathbf{x}^* = \mathbf{x}^*(a,\ t),\ \mathbf{u}^* \equiv \partial\mathbf{x}^*(a,\ t)/\partial t, \tag{2.2}$$

in which $\mathbf{x}^*$ is determined in $\overset{*}{\tau}_+$. The dependence of $\mathbf{x}$ and $\mathbf{x}^*$ on one and the same Lagrangian coordinates $\mathbf{a}$ indicates the establishment of correspondence between fluid particles in motion (2.1) and (2.2) $\mathbf{x}^*(\mathbf{a}, t)$, as well as $\mathbf{x}(\mathbf{a}, t)$, satisfy equations

$$\frac{\partial^2 x_m^*}{\partial t^2}\frac{\partial x_m^*}{\partial a_h} = -\frac{\partial p^*}{\partial a_h},\quad \det\left\|\frac{\partial x_i^*}{\partial a_k}\right\| = 1. \tag{2.3}$$

The following notations are used for simplicity: $(x_1, x_2)\equiv(x, y)$, $(U_1, U_2)\equiv(U, V)$, etc. Repeated indices with vectors imply summation. The Lagrangian displacement field ξ, which subsequently plays the crucial role, is determined by

$$\xi(\mathbf{a},\ t) \equiv \mathbf{x}^*(\mathbf{a},\ t) - \mathbf{x}(\mathbf{a},\ t). \tag{2.4}$$

Now, using the inverse of function (2.1) $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$, Eqs. (2.2)-(2.4) can be rewritten in terms of independent variables $\mathbf{x}$, t. Equations (2.4) gives the relation $\mathbf{x}^*(\mathbf{x}, t) = \mathbf{x} + \xi(\mathbf{x}, t)$, which means that the particle in the basic flow with Eulerian coordinate $\mathbf{x}$ from τ_+ has a coordinate $\mathbf{x}^*$ in the disturbed flow from $\overset{*}{\tau}_+$. It follows from the determination of velocity (2.2)

$$D[\mathbf{x} + \xi(\mathbf{x}, t)] = \mathbf{u}^*(\mathbf{x}^*, t),\quad D \equiv \frac{\partial}{\partial t} + U_\alpha \frac{\partial}{\partial x_\alpha},\qquad D\xi(\mathbf{x},\ t) = \mathbf{u}^*(\mathbf{x} + \xi,\ t) - \mathbf{U}(\mathbf{x},\ t) \equiv \delta\mathbf{u}(\mathbf{x},\ t), \tag{2.5}$$

where $\delta\mathbf{u}(\mathbf{x}, t)$ is the Lagrangian velocity increment, i.e., the difference between the velocity of the one and the same fluid particle in the disturbed and the undisturbed flows. The substitution of the variables in (2.3) gives

$$D^2\xi_i + \frac{\partial\xi_k}{\partial x_i}(A_k + D\xi_k) = -\frac{\partial\delta p}{\partial x_i},\quad \det\left\|\delta_{ik} + \frac{\partial\xi_i}{\partial x_k}\right\| = 1, \tag{2.6}$$

in which $\delta p(x, t)$ is the Lagrangian pressure increment such that $p^*(x^*, t)\equiv P(x, t) + \delta p(x, t)$; δ_{ik} is a unit matrix; $A_k = U_m \partial U_k/\partial x_m = -\partial P/\partial x_k$. Equation (2.6) should be supplemented with no-slip boundary conditions on $\partial\tau$ and matching conditions (1.5) with similar solutions from τ_-. Choosing the point $\mathbf{x}$ on R and using Eq. (2.5), the condition (1.5) can be written as

$$[\mathbf{u}^*(\mathbf{x} + \xi,\ t)]^* = [\delta\mathbf{u}(\mathbf{x},\ t)] = D[\xi(\mathbf{x},\ t)] = 0. \tag{2.7}$$

Integrating the last equality and using Lagrangian coordinates from the two regions of R give

$$[\xi] = [\eta] = 0,\ \xi = (\xi,\ \eta) \equiv (\xi_1,\ \xi_2). \tag{2.8}$$

Then, after specifying the initial conditions for the determination of functions $\xi_\pm (\mathbf{x}, t)$, we obtain an initial boundary-value problem in the regions with fixed boundaries $R_\pm$, R.

A linearized version of this problem appears as follows. The following equations are solved

$$D^2\xi_i + A_k \frac{\partial \xi_h}{\partial x_i} = -\frac{\partial \delta p}{\partial x_i}, \quad \frac{\partial \xi_k}{\partial x_k} = 0 \tag{2.9}$$

with boundary conditions

$$\xi \cdot \mathbf{n} = 0 \tag{2.10}$$

on $\partial\tau$ and matching conditions (2.8) on R.

The problem (2.8)-(2.10) can be reduced to a form that is identical with linearized Eulerian formulation. In order to achieve this, new fields $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ are introduced, and these are determined by the following equations

$$\delta \mathbf{u} = D\xi \equiv \mathbf{u} + (\xi\nabla)\mathbf{U}, \ \delta p \equiv p + (\xi\nabla)P. \tag{2.11}$$

The substitution of (2.11) in (2.9) leads to equations

$$Du + U_x u + U_y v = -p_x, \ Dv + V_x u + V_y v = -p_y, \quad u_x + v_y = 0, \ u = (u, v). \tag{2.12}$$

Eliminating p from (2.12) and using (1.2), we get an equation for $\omega \equiv v_x - u_y$:

$$D\omega + \Omega_x u + \Omega_y v = 0. \tag{2.13}$$

Equations (1.3), (2.10), and (2.11) lead to the boundary condition on $\partial\tau$

$$\mathbf{u} \cdot \mathbf{n} = 0. \tag{2.14}$$

The following conditions on R arise as a result of (2.7):

$$[\mathbf{u} + (\xi\nabla)\mathbf{U}] = 0. \tag{2.15}$$

It is convenient to introduce unit normal $\mathbf{v}$ and tangential $\boldsymbol{\sigma}$ vectors to the stream line (1.1) and Lagrangian displacement N along the normal to it:

$$Q\mathbf{v} = (-V, \ U), \ Q\boldsymbol{\sigma} = (U, \ V), \ N \equiv \xi \cdot \mathbf{v}. \tag{2.16}$$

Relations on R follow from (1.4), (2.8), and (2.15):

$$[\mathbf{u} \cdot \mathbf{v}] = [p] = 0, \ [\mathbf{u} \cdot \boldsymbol{\sigma}] = N[\Omega]. \tag{2.17}$$

The equation $D(V\xi - U\eta) = Vu - Uv_1$ is quite as easily verified and rewritten in the form

$$D(QN) = Q\mathbf{u} \cdot \mathbf{v} \tag{2.18}$$

and denotes linearized kinematic conditions.

Thus, in terms of (2.11), the problem of describing small disturbances consists in the solution of Eqs. (2.12) in fixed regions $\tau_\pm$ with boundary conditions (2.17), (2.18) on $R_\pm$ and R. In this type of description, the Lagrangian displacements exist explicitly only through N on R. Equations (2.12) structurally coincide with linearized equations for Eulerian disturbances, and the relation (2.15) coincides with (1.5), which is obtained as described before in the procedure for the "transposal" of conditions from R* on R. We note that the fields $\mathbf{u}$, p are actually Eulerian disturbances at those points $\mathbf{x}$ where the very concept of Eulerian disturbances has meaning. For clarification, it is sufficient to describe the determination of Eulerian velocity disturbances as a difference of velocities in the disturbed and basic flows at one and the same location $\mathbf{x}^*$ from $\tau_\pm^*$:

$$\mathbf{u}'(\mathbf{x}^*, \ t) \equiv \mathbf{u}^*(\mathbf{x}^*, \ t) - \mathbf{U}(\mathbf{x}^*) = \delta\mathbf{u}(\mathbf{x}, \ t) + \mathbf{U}(\mathbf{x}, \ t) - \mathbf{U}(\mathbf{x}^*, t), \tag{2.19}$$

where $\delta\mathbf{u}$ (2.5) and $\mathbf{x}^* = \mathbf{x} + \xi$ have been used. The expression (2.19) is based on the assumption that the point $\mathbf{x}^*$ belongs to τ_+. In this case the expansion of the function $\mathbf{U}(\mathbf{x} + \xi, t)$, $\mathbf{u}'(\mathbf{x} + \xi, t)$ in a series near the point $\mathbf{x}$ and subsequent linearization lead to the relation $\mathbf{u}' \equiv \delta\mathbf{u} - (\xi\nabla)\mathbf{U}$, which, when compared to (2.11), result in $\mathbf{u}' = \mathbf{u}$. If, however, the point $\mathbf{x}^* = \mathbf{x} + \xi$ is outside τ_+, then $\mathbf{u}$ and p cannot be traced as Eulerian disturbances, and their interpretation follows from Eqs. (2.11). In particular, as a result of this the discontinuity in the tangential component of $\mathbf{u}$ on R (2.17) will not appear as a discontinuity in velocity disturbance field.

As already mentioned, problems of the type formulated for the fields **u**, p are usually obtained directly by linearization in Eulerian coordinates [7, 8]. The need for paying attention to (2.11) arises only for detailed analysis of the meaning of the fields **u**, p and the relations they satisfy. Hence, the problem (2.12), (2.14), (2.17), and (2.18) is treated further as an independent problem, and the fields **u**, p, and ω are called disturbances in velocity, pressure, and vorticity respectively.

3. Integral of the Linear Problem. The following divergent relation is obtained as a result of Eqs. (1.2), (2.12), and (2.13)

$$DE/2 + (up + vA + U\varepsilon)_x + (vp - uA + V\varepsilon)_y = 0, \; E \equiv u^2 + v^2 + d\Psi\omega^2/d\Omega, \; A \equiv Vu - Uv, \; \varepsilon \equiv (u^2 + v^2)/2, \tag{3.1}$$

where it is assumed that $\Omega' \neq 0$. If $[\Omega] = [\Omega'] = 0$ in (1.4), then the conservation of the following functional results from (1.3), (2.14), and (3.1):

$$\int_\tau E dx dy = \text{const.} \tag{3.2}$$

This result is a reformulation of the expressions in [2-4]. If the known generalized Rayleigh criterion on inflection point $\Omega' \neq 0$ is satisfied in τ, then (3.2) results in flow stability in the rms sense. Extending the integral (3.2) to the problem with discontinuities Ω and Ω' leads to

$$I \equiv \int_\tau E dx dy - [\Omega] \int_R Q N^2 dl = \text{const}, \tag{3.3}$$

where τ implies sum of the integrals along τ_+ and τ_-; the positive direction for integration along R is based on the vector **U** on R. The proof for (3.3) is carried out by direct computation of the derivative dI/dt using (1.3), (2.14), (2.17), (2.18), and (3.1).

For the important class of flows with piecewise constant vorticity $\Omega' \equiv 0$, and hence E in (3.1) and (3.3) loses significance. In this case the integral for the linear problem can be obtained at the expense of narrowing the class of disturbances.

The vorticity equation (2.13) using (2.11) is brought to the form $D(\omega + \xi\Omega_x + \eta\Omega_y) = 0$. If the initial condition is chosen as

$$\omega = -\xi\Omega_x - \eta\Omega_y = \Omega' Q N, \tag{3.4}$$

then this equality will be satisfied for all t. The function N in (3.4) is the Lagrangian displacement (2.16) normal to any streamline of the flow (1.1), and not only to R. The equation (3.4) indicates the limitation of the class of disturbances to the so-called "equivortical" [4] characterized by the fact that the circulation is constant for each fluid particle and the vorticity field changes only due to the replacement of these particles. The interal (3.3) for this narrower class of disturbances remains valid, only E in (3.1) takes the following form in accordance with (3.4)

$$E = u^2 + v^2 + \Omega' Q^2 N^2. \tag{3.5}$$

When $\Omega \equiv 0$, (3.5) gives $E = u^2 + v^2$, and from (3.4) it follows that $\omega \equiv 0$. Thus, if e.g., in τ_+ there $\Omega = \Omega_+ = \text{const}$, then, assuming potential flow in τ_+^*, we again get the integral (3.3) in which we use the equation $E = u^2 + v^2$ while integrating along τ_+. If $\Omega = \Omega_- =$ const also in τ_-, then in both regions $\omega \equiv 0$, and (3.3) is reduced to the form

$$I = \int_\tau (u^2 + v^2)\, dx dy - [\Omega] \int_R Q N^2 dl = \text{const.} \tag{3.6}$$

4. General Stability Conditions. The effect of types of stable flows in the rms sense is then reduced to finding the sign convention for the quadratic form in the integrals (3.3) and (3.6), whose nature determines the stability. The determination of the deviation of the disturbed flow from the undisturbed flow, on the basis of the integrals I from the equation $I(t) \equiv$ const, leads to stability in the Liapunov sense: for any number $\varepsilon > 0$, there is another number $\delta > 0$ such that when $I(0) < \delta$, for all t, $I(t) < \varepsilon$. It is especially important to consider two cases here. Firstly, the presence of Lagrangian displacements N of the surface of discontinuity $[\Omega] \neq 0$ is a significant difference from stability conditions for flows with smooth vortex [2-4]. Secondly, it follows from (3.3) that the presence of weak discontinuities in circulation $[\Omega] = 0$, $[\Omega'] \neq 0$ leads to the same integral (3.2) that is obtained for smooth fields Ω, and, consequently, to the same stability criteria of [2-4].

The integral (3.3) is positive definite if in τ_+ and τ_-, $\Omega' > 0$, and on R, $[\Omega] < 0$. If a piecewise continuous function $\Omega(\Psi)$ is introduced in the entire region τ, then these requirements indicate that the circulation Ω should monotonically increase with Ψ. Similarly, the integral (3.6) is positive definite if the piecewise constant function $\Omega(\Psi)$ does not increase monotonically. In both cases the flow in a closed ring is stable in the above described sense.

At the same time the derivation of integrals (3.3) and (3.6) is based only on the presence of the divergent form (3.1) and boundary conditions (1.3), (2.14), and (2.17). Hence, Eqs. (3.3) and (3.6) are valid for flows (1.1) with practically any geometry, with any number and location of discontinuities R, and can be used as a proof of stability. However, it appears that if the flow is divided into a number of simply connected regions with closed streamlines, then the conditions for the monotonical increase of $\Omega(\Psi)$ identify an extremely narrow class of flows which do not include many problems of practical interest. For example, in the case of the known channel flow problem with separated outer flow and the formation of circulatory flow within the channel itself [1], integrals (3.3) and (3.6) are positively determined only when the outer flow is vortical with its circulation exceeding (in magnitude) the circulation in the circulatory region of the flow. If, however, the outer flow is irrotational, then the sign of the surface integral in (3.6) is always negative. The same is true for the plane analog of a vortex ring (for a vortex pair with finite core circulations) and for the plane analog of Hill's vortex.

Furthermore, in stable flows with monotonically increasing $\Omega(\Psi)$, disturbances N could increase near the stagnation point Q = 0. The important role of such points for specific flows has already been mentioned in [9].

5. Plane-parallel and Circular Flows. The most extensive class of sufficient conditions for stability can be obtained for flows with symmetry. For plane-parallel flow the region τ is determined by the limit $0 < y < H$, and the velocity field (1.1) has the form $U = U(y)$, $V \equiv 0$ with a continuous function U(y). The circulation $\Omega(y)$ is a piecewise continuous function. In the continuous segments $\Omega \equiv -U_y$ and at the final set of points $y = y_n (n = 1, 2, 3, \ldots, m)$, there are finite jumps $[\Omega]_n \neq 0$, which are conveniently determined here as $[\Omega]_n \equiv \Omega(y_n + 0) - \Omega(y_n - 0)$. The integral (3.3) takes the form

$$I = \sum_n \left(\int_{\tau_n} E\,dx\,dy + U_n [U_y]_n \int_{R_n} \eta^2 dx \right) = \text{const}, \qquad (5.1)$$
$$E = u^2 + v^2 + \omega^2 U / U_{yy}.$$

In order to obtain the integral of the type (3.6), it is necessary to assume $E = u^2 + v^2$ in (5.1). From (5.1) and Galilean invariance follows the conservation of the functionals

$$G = \sum_n \left(\int_{\tau_n} \frac{\omega^2}{U_{yy}}\,dx\,dy + [U_y]_n \int_{R_n} \eta^2 dx \right) = \text{const},$$
$$G_0 = \sum_n [U_y]_n \int_{R_n} \eta^2 dx = \text{const}. \qquad (5.2)$$

The integral G occurs when $U_{yy} \neq 0$. The stability of flows in the rms sense follows from its conversion in cases when the values of U_{yy} in all intervals of continuity τ_n and $[U_y]_n$ at all jumps in R_n have the same sign. The integral G_0 exists for the piecewise linear profile U(y) and irrotational disturbances $\omega = 0$. Its form makes it possible to determine the stability in cases where all discontinuities in $[U_y]$ have the same sign.

Thus, the generalization of Rayleigh's criterion [10] for the stability of plane-parallel flows with continuous U(y), $\Omega(y)$ to profiles with continuous U(y) but discontinuous $\Omega(y)$ is the requirement of monotonical variation (increase or decrease) in the function $\Omega(y)$.

Flows with circular streamlines are analyzed in a similar manner. In polar coordinate system r, θ, the flow region is defined by the circular ring $R_1 < r < R_2$. If U, V are the radial (r) and tangential (θ) velocity components, then the flow is determined by the expressions $U = U(r)$, $V \equiv 0$ with the continuous function U(r). The circulation $\Omega(r)$ isa piecewise continuous function. In the continuous regions $\Omega = (rR)_r / r$ and at the end points $r_n (n = 1, 2, 3, \ldots, m)$, there are finite jumps $[\Omega]_n \neq 0$, which are conveniently determined as $[\Omega]_n \equiv \Omega(r_n + 0) - \Omega(r_n - 0)$. The integral (3.3) takes the form

$$I = \sum_n \left(\int_{\tau_n} E\,r\,dr\,d\theta + r_n U_n [\Omega]_n \oint_{R_n} N^2 d\theta \right), \quad E = u^2 + v^2 + \omega^2 U / \Omega_r. \qquad (5.3)$$

In order to obtain an integral of the type (3.6), it is necessary to assume $E = u^2 + v^2$ in (5.3). Since the equations for plane flows of a homogeneous fluid and the no-slip boundary conditions at the circular walls are invariant to transformation into a rotational system of coordinates with arbitrary constant velocity [11], it follows from (5.3) that the condition for the conservation of functions is

$$J = \sum_n \left(\int_{\tau_n} \frac{\omega^2}{\Omega_r} r^2 dr d\theta + r_n^2 [\Omega]_n \oint_{R_n} N^2 d\theta \right), \quad J_0 = \sum_n r_n^2 [\Omega]_n \oint_{R_n} N^2 d\theta. \tag{5.4}$$

The integral J exists when $\Omega_r \neq 0$ and J_0 for piecewise constant functions $\Omega(r)$ and $\omega \equiv 0$. As in the previous case, stability of circular flows with monotonically varying $\Omega(r)$ follows from (5.4).

The well-known Kelvin's vortex belongs to this class of flows for which $U = \Omega_0 r$ when $0 < r < a$ and $U = \Omega_0 a^2/r$ when $a < r < \infty$. It follows from (5.4) that

$$\oint N^2 d\theta = \text{const}, \tag{5.5}$$

where the integral is taken along the boundary of the vortex core $r = a$. The equality (5.5) indicates stability of Kelvin's vortex relative to disturbances with $\omega = 0$. In [12] this result was found using spectral theory.

6. Integrals of Exact Solutions. For integrals G_0 (5.2) and J_0 (5.4), it is possible to construct nonlinear analogs which can be obtained from the integrals for the vortex momentum G^* and the moment of momentum J^*:

$$G^* \equiv \int_\tau \omega^* y dx dy = \text{const}, \quad J^* \equiv \int_\tau \omega^* r^3 dr d\theta = \text{const} \tag{6.1}$$

(ω^* is the total circulation).

It appears that for plane-parallel flow, the integral G_0 (5.2) is valid even in the case of nonlinear equations. In order to prove this conclusion, first consider separately the layer $\tau_n(y_{n-1} < y < y_n)$ with circulation Ω_n. The superposition of disturbances transforms τ_n to the curvilinear region $\tau_n^*(y_{n-1} + \eta_{n-1}(x, t) < y < y_n + \eta_n(x, t))$. For incompressible flow, the following equation is valid for any given n

$$\int \eta_n(x, t)\, dx = 0. \tag{6.2}$$

The contribution to the vortex momentum G^* (6.1) from the integration with respect to τ_n^* is

$$\Omega_n \int \left(\int_{y_{n-1}+\eta_{n-1}}^{y_n+\eta_n} y dy \right) dx = \frac{\Omega_n}{2} \int \{(y_n + \eta_n)^2 - (y_{n-1} + \eta_{n-1})^2\}\, dx.$$

Now, considering (6.2) and neglecting terms independent of time, we find that this contribution is proportional to the expression $\Omega_n \int (\eta_n^2 - \eta_{n-1}^2)\, dx$, whose sum across all layers leads to the equation $G_0 = \text{const}$ (5.2), but already on the strength of exact equations of motion. This means that for plane-parallel flows with piecewise linear velocity profile U(y), the sufficiency condition for nonlinear instability is that the function $\Omega(y)$ be monotonical.

The nonlinear analog of the integral J_0 (5.4) for circular flow with piecewise constant vorticity is constructed along the same lines. Firstly, the layer $\tau_n(r_{n-1} < r < r_n)$ with circulation Ω_n is independently analyzed. The superposition of disturbances transforms τ_n into a curvilinear region $\tau_n^*(r_{n-1} + N_{n-1}(\theta, t) < r < r_n + N_n(\theta, t))$. The following equation is valid for any n for any incompressible flow

$$\int_0^{2\pi} (2r_n N_n + N_n^2)\, d\theta = 0. \tag{6.3}$$

Computation of the contribution to J^* (6.1) from the integration along τ_n^* and summation of these contributions using (6.3) lead to

$$2(J^* - J_0^*) = \sum_n [\Omega]_n \int_0^{2\pi} N_n^2 (2r_n + N_n)^2\, d\theta, \tag{6.4}$$

where J_0^* is the value of the integral J^* in the absence of disturbances $N_n \equiv 0$; $[\Omega]_n = \Omega(r_n + 0) - \Omega(r_n - 0)$. Linearization of the equations of motion reduces (6.4) to (5.4). Nonlinear stability follows from the inequality $2(J^* - J_0^*) \geqslant \sum_n [\Omega]_n r_n^2 \int_0^{2\pi} N_n^2 d\theta$, which results from (6.4) and the inequality $|N_n| < r_n$, which is the result of the definition of N. Thus, the monotonical behavior of $\Omega(r)$ is a sufficient condition for nonlinear stability of flows with circular streamlines and piecewise constant circulation. In particular, Kelvin's vortex is stable with respect to finite disturbances.

In conclusion, the following points are made.

1. The proof of stability for the class of plane disturbances has limited physical significance. Here it is possible to say only that the mechanism of the generation of plane disturbances is not effective, and flow instability, if it exists, is three-dimensional in nature.

2. All the results obtained here are applicable to axisymmetric flows and flows with helical geometry. Corresponding conclusions on stability are those given in [13, 14].

3. The practically important additional class of stable flows will be expressed as cases of negative definite integrals I (3.3) and (3.6). This class includes, e.g., channel flow, Hill's vortex, and Kirchoff's elliptic vortex. However, the value of the constant in the determination of the negative definitenes depends on the flow geometry. Its computation is an independently serious problem.

4. A variational principle, similar to that presented in [15], lies as the basis of the conclusions made above on stability. In obtaining it, it is necessary to remember that the condition of "equivoricity" is formulated separately for each fluid region where the vorticity varies continuously. On the basis of (2.8), functions transforming $\tau_\pm^*$ on $\tau_\pm$ can be chosen to be identical on R. Interals I, G, and J arise in this case as secondary variations in energy, momentum, and moment of momentum (6.1). At the same time, it is necessary to emphasize that in the present work, the linear stability is investigated in relation to arbitrary disturbances, and not only for "equivortices."

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